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STEADY-STATE ANALYSIS OF DELAYS IN OPEN AND CLOSED NETWORKS

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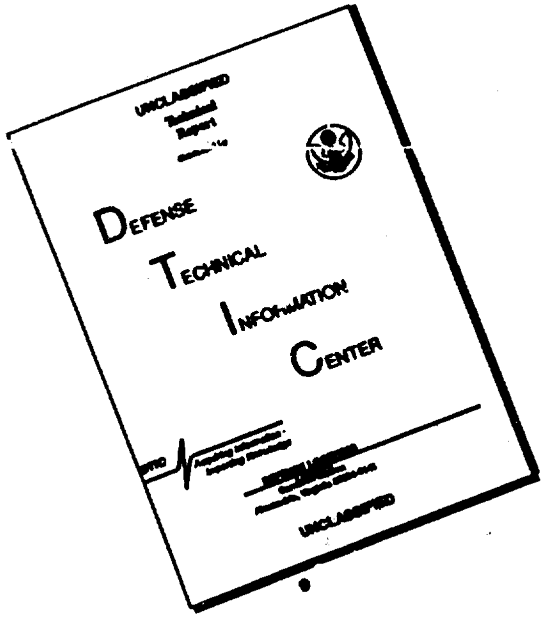

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ABSTRACT

The steady-state analysis of open or closed networks consisting of n nodes interconnected by links over which mass flows is the subject of this report. Formulas are derived which yield the equilibrium mass balance at each node in terms of a set of specified parameters which characterize both nodes and links. The solution is then generalized to the case of a k-priority network of interconnected nodes, either open or closed. k-priority systems are solved recursively and formulas are given which determine delays at each node for a fixed priority mass element in terms of known parameters.

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INTRODUCTION

A dynamic model of a materials balance consisting of a system of simultaneous equations has three possible modes of behavior as time increases. There may be unlimited increase or decrease of a variable; there may be attained constant levels of the variables not depending upon time; or there may be oscillatory solutions. Thus when the derivatives of the dependent variables are set equal to zero, the roots of the resulting system of equations identify the combinations of values of the dependent variables within which the steady-state combinations are a subset. By steady state one means either the case in which the dependent variables attain constant values independent of time or the case in which they yield fixed temporal averages. The linear steady-state network model of an open system referred to here can best be visualized in terms of a network flow diagram containing several nodes or points of accumulation of "species," i.e., substances of some kind connected by linkages over which material transfer occurs that is governed by physical laws or man-made control rules or both. A basic governing principle in the construction of the model which is true for deterministic cases and for a wide variety of cases in which the flows are governed by stochastic rules [1] is the following:

total means input rate to cell i =

$$= \frac{Q_i}{D_i} = \frac{\text{volume of cell i}}{\text{mean delay time in cell i}} =$$

= total mean output rate from cell i. (1)

A set of external sources $\{e_i\}$ and sinks $\{d_i\}$ identify those inputs and outputs of substances which have the effect of forcing exchanges among cells in such a way that a set of constant flows across nodes results. In a steady-state condition the sum $\sum e_i$ necessarily equals the sum $\sum d_i$. One anticipates that this condition can be satisfied in which different combinations of mean flow rates across nodes are possible and in fact it is shown below that one can set independently the values of mean flow rate across a fixed subset of nodes while satisfying the side condition $\sum e_i = \sum d_i$. The solution one obtains is in terms of the constant ratios

$$r_i = \frac{Q_i}{D_i} \tag{2}$$

which are the limiting values obtained by averaging the input rate to node i over time from 0 to + ∞. Knowing the ratios r_i and either the mean residence time D_i or cell volume D_i the remaining quantitiy can be determined. The model can be used for identifying feasible sets of external demands upon the network as well as obtaining specific solutions. In application the nodes might represent centers in which traffic of some kind arrives and departs after the occurrence of certain events at the node. The sinks might also represent demand points for a commodity such as water over a geographic area.

EQUATIONS OF MASS BALANCE

An open system differs from a closed system in that there are exogeneous sources which feed mass into the system at one or more nodes at specified rates. In addition there are exogeneous sinks into which mass flows from certain nodes at specified rates. If the system is in a steady-state condition, the mass input rate must equal the mass output rate. A closed system is obtained if this value is identically zero. A specification of the exogeneous inputs e_i and outputs d_i does not determine the flow rates $\{u_i\}$ into nodes from strictly *internal circulations* within the system. It does, however, restrict the set of possible combinations of u_i's and hence r_i's that will satisfy the mass balance equations. By definition, one has in steady-state that

$$\frac{Q_i}{D_i} = r_i + e_i = u_i + d_i \equiv r_i \tag{3}$$

where

d_i = mass flow rate to an exogeneous sink from node i $(i = 1,\ldots,n)$.

e_i = mass flow rate into node i from an exogeneous source. $(i = 1,..,n)$

u_i = non-negative mass flow rate out of node i to other nodes in the network. $(i = 1,\ldots,n)$

Although the average through puts $\{r_i\}$ must exist if the system has a steady state, they do not necessarily represent parameters explicit in the formulation of the time dependent physical dynamics of the materials balance equations. They may be used to construct steady-state equations of balance *in lieu of* differential equations governing the temporal dynamics of the system. In this case,

one may specify independently certain *directional coefficients* which are needed in order to solve for the $\{r_i\}$ which yields *rate information* (not accumulations). Let

$$P = [p_{ij}] \tag{4}$$

where

p_{ij} = fraction of mass leaving node i that arrives at node j. $(i,j = 1,\ldots,n)$.

In some applications there may be limits upon the rates at which mass can enter or leave or both enter and leave a node. Let

$\overline{a}_i$ = maximum allowable arrival rate of mass into node i from combined sources. $(i = 1,\ldots,n)$ (5)

$\underline{a}_i$ = minimum allowable arrival rate of mass into node i from combined sources. $(i = 1,\ldots,n)$ (6)

$\overline{b}_i$ = maximum allowable departure rate of mass from node i. $(i = 1,\ldots,n)$ (7)

$\underline{b}_i$ = minimum allowable departure rate of mass from node i. $i = (1,\ldots,n)$

Note that $\underline{a}_i = u_i + e_i = \overline{a}_i$ and $\underline{b}_i = u_i + d_i = \overline{b}_i$ $(i - 1,\ldots,n)$

For the i-th node, therefore, one has under steady-state condition that

$$\sum_{j=1}^{M} u_j\, p_{ji} + e_i = u_i + d_i, \quad (i=1,\ldots,n) \tag{8}$$

or, since

$$Q_i = (u_i + d_i)D_i,$$

$$\sum_j \left(\frac{Q_j}{D_j} - d_j\right) p_{ji} + e_i = \frac{Q_i}{D_i}$$

or,

$$\sum_{j=1}^{M} \frac{Q_j}{D_j} p_{ji} - \sum_{j=1}^{M} d_j p_{ji} + e_i - \frac{Q_i}{D_i} = 0 \qquad (9)$$

$$(i = 1,..,n)$$

Equations (9) can be written as a single $n \times n$ matrix equation in cases where the p_{ij}'s are specified in advance.

$$\begin{bmatrix} -(1 - p_{11}) & p_{21} & --- & p_{n1} \\ p_{12} & -(1-p_{22}) & --- & p_{n2} \\ \vdots & \vdots & & \vdots \\ p_{1n} & p_{2n} & --- & -(1-p_{nn}) \end{bmatrix} \begin{bmatrix} \frac{Q_1}{D_1} \\ \frac{Q_2}{D_2} \\ \vdots \\ \frac{Q_n}{D_n} \end{bmatrix} = \begin{bmatrix} \sum d_j p_{j1} - e_1 \\ \sum d_j p_{j2} - e_2 \\ \vdots \\ \sum d_j p_{jn} - e_n \end{bmatrix}$$

or,

$$PR = C \qquad (10)$$

where R is the $n \times 1$ column vector of ratios Q_i/D_i and C is the column vector on the right hand side. If one is dealing with a closed system in which the right hand side of (10) is identically zero then equation (10) is replaced by

$$
\begin{bmatrix}
-(1-p_{11}) & p_{21} & \text{-----} & p_{n1} \\
p_{12} & -(1-p_{22}) & \text{-----} & p_{n2} \\
\vdots & \vdots & & \vdots \\
p_{1n} & p_{2n} & & -(1-p_{nn}) \\
1 & 1 & \text{-----} & 1
\end{bmatrix}
\begin{bmatrix}
\frac{Q_1}{D_1} \\
\frac{Q_2}{D_2} \\
\vdots \\
\frac{Q_n}{D_n}
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
Q
\end{bmatrix}
\qquad (11)
$$

where

Q = total mass conserved in system.

In applications involving materials handling systems or certain traffic systems the p_{ij}'s may be fixed in advance or it may be clear as to which combinations of p_{ij}'s are permissible. In other applications where one is dealing with large volumes of a mass such as lake water or liquid moving through a porous medium the p_{ij}'s may be unknown but could be determined as part of the solution in which case one is dealing with $n^2 + n$ unknowns. The number of equations in this case depends upon boundary conditions governing the p_{ij} and other physical considerations. This paper deals only with the case in which the p_{ij}'s are known or can be specified parametically.

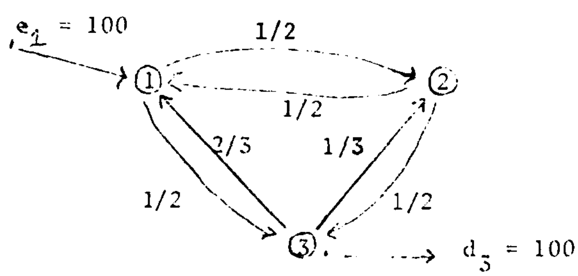
RANK OF THE P-MATRIX

It is easy to show that the n-th row of the P matrix in equation (10) is a linear combination of the first (n-1) rows the result being that its rank is at most n-1. Thus, at least one rate $\frac{Q_k}{D_k}$ (for some k) must be specified independently implying that many different internal circulation rates are possible while the external inputs $\{e_i\}$ and outputs $\{d_i\}$ remain fixed. However, $\frac{Q_k}{D_k} = r_k$ may be limited by restrictions (5), (6), and (7), i.e.,

$$\max\,[o,\ \underline{a}_k,\ \underline{b}_k] \leq r_k = u_k + d_k = \frac{Q_k}{D_k} \leq \min\,[\bar{a}_k,\ \bar{b}_k] \tag{12}$$

$$\text{and } o \leq u_k. \qquad\qquad (k = 1, .., n)$$

Consider the three node network specified below. Assume an exogeneous source at node 1 with $e_1 = 100$ and an exogeneous sink at node 3 with $d_3 = 100$. All other e_i and d_i are assumed to be zero.



$D_1 = 1,\ D_2 = 2,\ D_3 = 3$

One has for the balance equation

$$\begin{bmatrix} -1 & \frac{1}{4} & \frac{2}{9} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{9} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} 100 \cdot \frac{2}{3} - 100 \\ 100 \cdot \frac{1}{3} - 0 \\ 0 - 0 \end{bmatrix} \tag{13}$$

Upon solving (13) one finds

$$Q_1 = 22 + .37\ Q_3$$

$$Q_2 = -46 + .60\ Q_3 \qquad (14)$$

In other words the system has one degree of freedom but Q_3 is constrained by inequality (12). Suppose further that $\underline{a}_3 = \underline{b}_3 = 0$ and $\overline{a}_k = \overline{b}_k \to \infty$ so that (12) reduces to

$$0 \leqq u_3 + 100 = \frac{Q_3}{3}$$
$$0 \leqq u_3$$

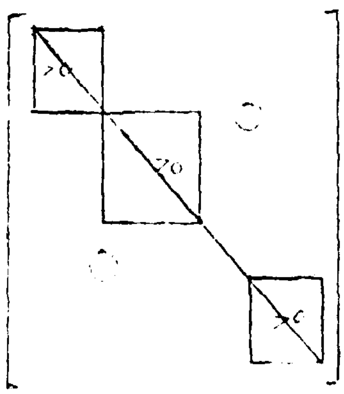
where upon

$$Q_3 \geqq 300.$$

If $Q_3 = 300$ then $u_3 = 0$ meaning that node 3 acts as a sink relative to internal circulations within the system. If one sets $Q_3 > 300$ then $u_3 > 0$ and node 3 no longer behaves as a sink. It is also clear that increasing Q_3 also increases the internal flow rate between other nodes.

It is apparent that provided the P matrix cannot be block-diagonalized to the form

P =



(15)

meaning that all nodes "communicate," the rank of P is $n-1$. If, in contrast, one can subdivide P as shown in (15) one is dealing with independent subnetworks which may be analyzed separately. The rank of P in this case is $n-\ell$ where ℓ is the number of blocks on the diagonal.

MEAN DELAY IN THE SYSTEM

The mean delay (mean residence time) at a node is related to mean flow across the node and node volume according to equation (3). The values of any two of the quantities u_i, Q_i, or D_i determines the third which holds for deterministic systems and for many stochastic systems [1, 2]. For present purposes it is assumed that mean delays at nodes are known or can be computed once the u_i's are known. The mean delay of an element of mass in the system can now be computed when the rule governing the transfer of mass among nodes is known. Little more can be said unless one assumes that mass elements move within the system at random so that at a particular node each element has an equal chance of being selected to move in a particular "direction." In this case the theory of first passage times for Markov chains applies and the mean delay of a unit in the system can again be computed in a straight forward manner. The computations are reviewed below.

Let

M_{ij} = mean delay encountered by a mass element moving from node i to node j. $(i,j = 1,..,n;\ i \neq j)$ (16)

The value of M_{ij} can be determined in the following way which is a straight forward application of the theory of first passage times for Markov chains. Denote by P^T the transpose of the P matrix. Let j denote the node to which a mass element is to be transported. Redefine the p_{ji}'s corresponding to node j as (1, 0, ---, 0) so that once a mass element enters node j it remains there. Reposition this vector as the first row in P^T so that the revised form of P^T, call it $\overline{P}^T$ is

$$\overline{P}^T = \begin{bmatrix} 1 & 0 & --- & 0 \\ P_{21} & P_{22} & --- & P_{2N} \\ P_{n1} & P_{n2} & --- & P_{nn} \end{bmatrix} \tag{17}$$

The only requirement on the order of the remaining rows and columns is that their order agree. That is, row order must equal column order. (In many cases $p_{ii} = 0$). Denote by S the submatrix obtained from $\overline{P}^T$ by deleting the first row and first column. Let $[m_{ij}]$ be the matrix defined by

$$[m_{ij}] = [I - S]^{-1} \tag{18}$$

Then

$$M_{ij} = \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} m_{i\ell}\, D_\ell \tag{19}$$

Example. Continuing with the previous example, and assuming that $p_{ii}=0$, set $j = 3$ so that

$$\bar{P}^T = \begin{array}{c} \\ 3 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array} \tag{20}$$

and

$$S = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} \tag{21}$$

Then

$$I - S = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \tag{22}$$

and

$$[I - S]^{-1} = \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \tag{23}$$

The mean delay encountered by a mass element in reaching node 3 from node 1 is therefore

$$M_{13} = m_{11} D_1 + m_{12} D_2 =$$

$$= \frac{4}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{8}{3} \tag{24}$$

Similarly

$$M_{23} = m_{21} D_1 + m_{22} D_2 =$$

$$= \frac{2}{3} \cdot 1 + \frac{4}{3} \cdot 2 = \frac{10}{3} \tag{25}$$

MULTIPLE PRIORITY SYSTEMS

Multiple priority systems occur when mass elements are tagged in a way which identifies them by the mean delay they encountered at a node. Whenever one "species" takes precedence over another at a given node in such a way that it encounters a shorter time delay, it is said to have a higher priority than the other species. The only complicating factor entering into the analysis of the multiple priority system is the computation of the mean delays for the lower priority elements since they now depend upon the mean delays of the higher priority elements.

BALANCE IN A 2-PRIORITY SYSTEM

It is assumed that the network contains two "species" of mass which flow through nodes and over links in the manner described above. The difference here is that one species takes precedent over the other in flowing through any node. The "low priority" species flows through a node only when there is no "high priority" mass at that node. Denoting the high and low priorities by 1 and 2, respectively, one defines the following symbols.

Let

$$D_{1i} = \text{mean delay at node } i \text{ of priority 1 mass elements} \qquad (26)$$

$$(i = 1,\ldots,n).$$

$$D_{2i} = \text{mean delay at node } i \text{ of priority 2 mass elements} \qquad (27)$$

$$(i = 1,\ldots,n)$$

Q_{1i} = quantity of priority 1 mass present at node i under equilibrium conditions (28)

$$(i = 1,\ldots,n)$$

Q_{2i} = quantity of priority 2 mass present at node i under equilibrium conditions (29)

$$(i = 1,\ldots,n)$$

Further, let

r_{1i} and r_{2i} denote the average output rates of mass of priorities 1 and 2, respectively, from node i $(i = 1,..,n)$.

Let

p_{1ij} = proportion of priority 1 mass leaving node i that is directed to node j,

p_{2ij} = proportion of priority 2 mass leaving node i that is directed to node j,

and let

$$\rho_{ki} = \frac{r_{ki}}{\bar{b}_{ki}} \qquad (30)$$

$$(k = 1, 2; i = 1,\ldots,n).$$

where

$\overline{b}_{ki}$ = maximum allowable departure rate of k-th priority mass from i-th node.

The coefficient ρ_{ki} always lies between 0 and 1 and is called the *load factor* for mass of priority k and the i-th node. This means the priority may change for a fixed mass element at different nodes.

The following assumptions are made concerning the behavior of the system containing mass of both high and low priority.

(1) The analysis with respect to priority 1 mass remains unchanged from that given above since it does not recognize the existence of priority 2 mass.

(2) If priority 1 mass is removed from the system, then priority 2 mass is the only priority present and the mass balance analysis is the same as that given above. In this case, the D_{2i} can be independently specified since there is, in fact, a single priority to be analyzed. In such a circumstance denote D_{2i} by the symbol D^*_{2i} where

D^*_{2i} = mean delay of priority 2 mass at node i when all priority 1 mass is absent from the system. (31)

(3) When both priorities of mass are present the mass balance equations for priority 2 follow those of priority 1 after the delays D_{2i} have been determined.

The delay of a priority 2 mass element at a node is the sum of 1) its delay D^*_{2i} when no priority 1 elements are at node i; 2) the delay D_{1i} due to the presence of priority 1 mass elements; 3) an additional delay generated by the arrival of priority 1 mass elements during the intervals D^*_{2i} and D_{1i} which is

$$\rho_{1i}\ (D^*_{2i} + D_{1i});$$

an additional delay generated by the arrival of priority 1 mass elements during the interval defined by $\rho_{1i}\ (D^*_{2i} + D_{1i})$ which is

$$\rho^2_{1i}\ (D^*_{2i} + D_{1i});$$

4) and so on.

When summed, one has that

$$D_{2i} = \sum_{j=0}^{\infty} \rho^j_{1i}\ (D^*_{2i} + D_{1i})$$

$$= (D^*_{2i} + D_{1i})\ \frac{1}{1-\rho_{1i}} \tag{52}$$

Note that all quantities on the right hand side of (52) are known. ρ_{1i} is obtained from the equation

$$\rho_{1i} = \frac{Q_{1i} D^{-1}_{1i}}{\bar{b}_{1i}} = \frac{r_{1i}}{\bar{b}_{1i}} \tag{53}$$

An equation analogous to (10) is now solved in which all parameters and variables pertain to priority-2 mass. Upon solution one obtains the ratios $r_{2i} = \frac{Q_{2i}}{D_{2i}}$.

From (32) one obtains D_{2i} and thus Q_{2i} can be computed. The essential point is that the mean delays for lower priorities cannot be specified independently as in the case of the priority-1 elements.

BALANCE IN A K-PRIORITY SYSTEM

One can now analyze closed or open networks having k priorities of mass units. Assuming that lower priority units do not affect the behavior of higher priority units, one solves for $(Q_{11},\ldots,Q_{1n})$, $(Q_{21},\ldots,Q_{2n})$, ..., $(Q_{k1},\ldots,Q_{kn})$ respectively. The solution of the mass balance equations for the i-th priority requires the previous (i-1) solutions. Definitions that correspond to (26) through (30) are made for each priority. The only real complexity that enters is the determination of D_{ji} which is the delay of a mass element of the j-th priority at node i. One can show that D_{ji} is given by the formula

$$\begin{aligned} D_{ji} &= D^*_{ji} + D_{j-1,i} + \cdots + D_{2i} + D_{1i} + \\ &+ \sum_{\lambda=1}^{\infty} (\sigma_{j-1,i})^{\lambda} \left[D^*_{ji} + D_{j-1,i} + \cdots + D_{2i} + D_{1i} \right] = \\ &= \frac{1}{1-\sigma_{j-1,i}} \left[D^*_{ji} + D_{j-1,i} + \cdots + D_{2i} + D_{1i} \right] \end{aligned} \tag{34}$$

where

$$\sigma_{j-1,i} = \rho_{1i} + \rho_{2i} + \cdots + \rho_{j-1,i}$$

$$(j = 2,3,\ldots,k) \quad (i = 1,2,\ldots,n).$$

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